Lecture 33

10.2 - Calculus with Parametric Curves

Suppose C is a parametric curve described by the parametric equations

$$x = f(t), \quad y = g(t).$$

Using the chain rule, we can find $\frac{dy}{dx}$ & $\frac{dx}{dy}$ in terms of $\frac{dx}{dt}$ & $\frac{dy}{dt}$:

$\boxed{\frac{dy}{dx}}$ $\quad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

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As usual, $\frac{dy}{dx}$ will give us the slope of the tangent line to the curve at a given point. The descriptions of $\frac{dy}{dx}$ & $\frac{dx}{dy}$ in terms of $\frac{dx}{dt}$ & $\frac{dy}{dt}$ can help us to find vertical and horizontal tangents:

Recall that a curve has a <u>horizontal tangent</u> when $\frac{dy}{dx}=0$. Using the above, we see this is the same as

$$\frac{dy}{dt}=0 \qquad \left(\frac{dx}{dt}\neq 0\right)$$

A <u>vertical tangent</u> happens when $\frac{dy}{dx}=\infty$ or $\frac{dx}{dy}=0$. Using parametric equations, this becomes:

$$\frac{dx}{dt}=0 \qquad \left(\frac{dy}{dt}\neq 0\right)$$

We can use the chain rule again to find second derivatives:

$$\boxed{\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}$$

which is valid as long as:

$$\frac{dx}{dt}\neq 0.$$

This lets us find where parametric curves are concave up or down at.

Ex: Consider the curve with parametric equations

$$x = t^3 - 3t, \quad y = t^2 - 3$$

ⓐ Find the points on the curve where the tangent is vertical.

$\frac{dx}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$

$\frac{dy}{dt} = 2t = 0$ when $t = 0$

Vertical tangents @ $t = \pm 1$

$\underline{t=-1}$ $(x,y) = (-1+3, 1-3) = (2,-2)$

$\underline{t=1}$ $(x,y) = (1-3, 1-3) = (-2,-2)$

ⓑ At what points is the tangent horizontal?

Horizontal tangent @ $t=0$

$\underline{t=0}$: $(x,y) = (0-0, 0-3) = (0,-3)$.

ⓒ Find an equation for the tangent line at $t=2$.

$\underline{@\, t=2}$: $(x,y) = (8-6, 4-3) = (2,1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}(@\, t=2) = \frac{-4}{12-3} = \frac{-4}{9}$$

Tangent line: $\boxed{y - 1 = \frac{-4}{9}(x-2)}$

(d) For what t-values is the graph concave up? concave down?

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{2t}{3t^2-3}\right)}{3t^2-3} = \frac{\frac{2(3t^2-3)-6t(2t)}{(3t^2-3)^2}}{3t^2-3} = \frac{-6t^2-6}{(3t^2-3)^3}$$

$-6t^2-6$ is always negative, so sign of $\frac{d^2y}{dx^2}$ depends on $3t^2-3$:

$\boxed{\text{concave up}}$ $\frac{d^2y}{dx^2}>0 \Rightarrow 3t^2-3<0 \Rightarrow t^2<1 \Leftrightarrow (-1,1)$

$\boxed{\text{concave down}}$ $\frac{d^2y}{dx^2}<0 \Rightarrow 3t^2-3>0 \Rightarrow t^2>1 \Leftrightarrow (-\infty,-1)\cup(1,\infty)$

Parametric curves can pass through the same point more than once... an not necessarily in the same way.

Ex: Find the slope of both tangent lines to

$$x=t\cos t,\quad y=t\sin t,\quad -\pi\le t\le\pi$$

at the point $(0,\frac{\pi}{2})$.

The curve passes through $(0,\frac{\pi}{2})$ when $t=\pm\frac{\pi}{2}$.

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos t - t\sin t}{\sin t + t\cos t}$$

$\boxed{@\, t=-\pi/2}$

$$\frac{dy}{dx}=\frac{0+\frac{\pi}{2}(-1)}{(-1-0)}=\boxed{\frac{\pi}{2}}$$

$\boxed{@\, t=\frac{\pi}{2}}$

$$\frac{dy}{dx}=\frac{0-\frac{\pi}{2}(1)}{1+0}=\boxed{\frac{-\pi}{2}}$$

Area Under a Curve

We know from calc I that the area under a curve $y = F(x)$, $a \le x \le b$, and $F(x) > 0$ is given by $\int_a^b F(x)\,dx$ ...

but sometimes this integral is tedious or impossible to compute. Suppose we parametrize this curve by

$$x = f(t),\ y = g(t),\ \alpha \le t \le \beta$$

Then, since $y = F(x)$

$$\int_a^b F(x)\,dx = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,d(f(t))$$

$$= \int_\alpha^\beta g(t) f'(t)\,dt$$

*It might be necessary to integrate from β to α... it depends on which direction the curve is parametrized in.

Area is given by

$$\int_\alpha^\beta g(t) f'(t)\,dt \quad \text{or} \quad \int_\beta^\alpha g(t) f'(t)\,dt$$

Ex: Find the area under the curve

$$x = 2\cos t,\ y = 3\sin t,\ 0 \le t \le \frac{\pi}{2}$$

$$\text{Area} = \int_0^{\pi/2} (3\sin t)\, d(2\cos t) = \int_0^{\pi/2} -6\sin^2 t\, dt$$

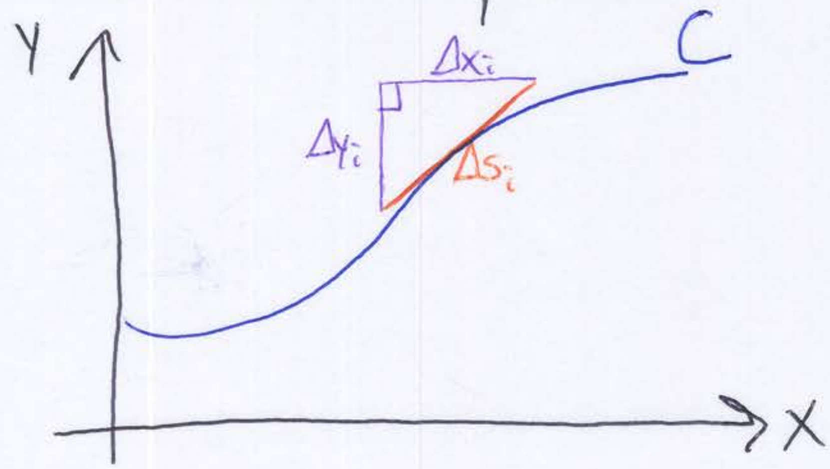
$$= -3\int_0^{\pi/2} (1-\cos 2t)\, dt = -3\left(t - \tfrac{1}{2}\sin 2t\right)\Big|_0^{\pi/2}$$

$$= -3\left[\left(\tfrac{\pi}{2}-0\right)-(0-0)\right] = -3 \ : \text{negative}$$

Needed to integrate from $\pi/2$ to 0 : $\boxed{\text{Area} = 3}$

Arc Length

Recall from chapter 8:



$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

So, the length of the curve C is given by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \int_C ds$$

If C has parametric equations

$$x = f(t), \ y = g(t), \ a \le t \le b$$

then

$$L = \int_C ds = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b \sqrt{[d(f(t))]^2 + [d(g(t))]^2}$$

$$= \int_a^b \sqrt{[f'(t)\,dt]^2 + [g'(t)\,dt]^2}$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

There are some conditions need for this though:

1) $f'(t)$ & $g'(t)$ are continuous on $[a,b]$

2) The curve C is traversed <u>exactly once</u> as t increases from a to b.

Ex: Show that the circumference of a circle of radius r is indeed $2\pi r$.

Parametrization of circle: $x = r\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi.$

$$L = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

Ex: Find the arclength of the circle defined by

$$x = \cos 2t,\ y = \sin 2t,\ 0 \le t \le 2\pi.$$

What is the significance of this?

$$L = \int_0^{2\pi} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}\, dt = \int_0^{2\pi} 2\, dt = 4\pi.$$

This parametrization goes twice around the circle, so we get the distance traveled, not the actual length of the curve.